Jai-chan Hwang[(a)] and Hyerim Noh[(b)]
[(a)] *Department of Astronomy and Atmospheric Sciences, Kyungpook National University, Taegu, Korea*
[(b)] *Korea Astronomy Observatory, Taejon, Korea*
(March 22, 2002)

If the linear perturbation theory is valid through the bounce, the surviving fluctuations from the ekpyrotic scenario (cyclic one as well) should have very blue spectra with suppressed amplitude for the scalar-type structure. We derive the *same* (and consistent) result using the curvature perturbation in the uniform-field gauge and in the zero-shear gauge. Previously, Khoury *et al.* interpreted results from the latter gauge condition incorrectly and claimed the scale-invariant spectrum, thus generating controversy in the literature. We also correct similar errors in the literature based on wrong mode identification. No matching condition is needed for the derivation.

PACS numbers: 04.20.Dw, 98.80-k, 98.80.Cq, 98.80.Hw

I. INTRODUCTION

The issue of scalar-type structure generated in the recently proposed ekpyrotic scenario is shrouded with controversies by two opposing camps [1,2] and [3–8]. The main point of [1] is that the dominating solution viewed in the zero-shear hypersurface (gauge) in the collapsing phase happens to show a scale-invariant spectrum. However, this mode was identified in [4,5] as a transient mode in the subsequent expanding phase, thus uninteresting. In this work we wish to add some additional points to [5]. We will show that the same blue spectrum is generated even in the zero-shear gauge by identifying the mode relevant in the later expanding phase. Apparently, the same final observable spectrum should be derived independently of the gauge conditions used, and our result confirms it. We also point out that the possible scale-invariant spectra and others argued in [9,10] are errored by identifying wrong modes which are transient in the expanding phase, thus uninteresting.

In a single component fluid or field, the scalar-type perturbation is described by a second-order differential equation with two solutions (modes). In the large-scale limit (to be defined later) we can often derive a general asymptotic solution with two modes, see eq. (7). In an expanding phase we can identify clearly which ones are relatively growing (C-mode) and decaying (d-mode). If the initial condition is imposed at some early expanding epoch the decaying mode is transient in time, and naturally we are only interested in the relatively growing mode. If we introduce a collapsing phase before the early big-bang phase, however, the conventional growing and decaying classification can be often reversed. Still, if the large-scale conditions are met (and, of course, *if* the linear theory as well as the classical gravity are intact), the general solutions in eq. (7) remain valid throughout the transition. Thus, in our observational perspective located in expanding phase we are interested in the initial condition imposed on the C-mode, even if it *was* subdominating (relatively decaying) compared with the other mode when the initial condition was imposed. Although we made this point clear in [5], in this work we will reinforce it by deriving concretely the C-mode initial conditions coming from the quantum vacuum fluctuations in the two gauge conditions used previously. In this way, we hope we could clear some of the controversies concerning ekpyrotic scenario and others in the literature. §II and III are reviews. §IV contains our main results with discussions in §V.

II. BASIC EQUATIONS AND GENERAL LARGE-SCALE SOLUTIONS

We consider the scalar-type perturbation in a flat Friedmann world model supported by a minimally coupled scalar field. Our metric convention follows Bardeen's in [11]

$$ds^2 = -a^2(1+2\alpha)d\eta^2 - 2a^2\beta_{,\alpha}d\eta dx^\alpha + a^2\left[g^{(3)}_{\alpha\beta}(1+2\varphi) + 2\gamma_{,\alpha|\beta}\right]dx^\alpha dx^\beta, \quad (1)$$

and $\chi \equiv a(\beta + a\dot\gamma)$; an overdot and a prime indicate time derivatives based on t and η, respectively, with $dt \equiv a d\eta$. The background is described by

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot\phi^2 + V\right), \quad \ddot\phi + 3H\dot\phi + V_{,\phi} = 0, \quad (2)$$

where $H \equiv \frac{\dot a}{a}$. The basic perturbation equations are [12]:

$$u = -\frac{4\pi G z}{k^2}\left(\frac{v}{z}\right)', \quad v = \frac{1}{4\pi G z}(zu)', \quad (3)$$

$$v'' + \left[k^2 - \frac{z''}{z}\right]v = 0, \quad u'' + \left[k^2 - \frac{(1/z)''}{1/z}\right]u = 0, \quad (4)$$

where $z \equiv a\dot\phi/H$, and

$$v \equiv a\delta\phi_\varphi, \quad \varphi_{\delta\phi} \equiv \varphi - (H/\dot\phi)\delta\phi \equiv -(H/\dot\phi)\delta\phi_\varphi,$$
$$u \equiv -\varphi_\chi/\dot\phi, \quad \varphi_\chi \equiv \varphi - H\chi. \quad (5)$$

$\varphi_{\delta\phi}$ and φ_χ are gauge-invariant combinations which are equivalent to the curvature perturbation (φ) in the uniform-field gauge ($\delta\phi \equiv 0$) and in the zero-shear gauge ($\chi \equiv 0$), respectively [11]. The perturbed action was derived in [12]

$$\delta^2 S = \frac{1}{2}\int \left(v'^2 - v^{|\alpha} v_{,\alpha} + \frac{z''}{z} v^2 \right) d^3x d\eta. \qquad (6)$$

In the large-scale limit, meaning for negligible k^2 terms, eq. (4) has general solutions [7]

$$\varphi_{\delta\phi}(k,\eta) = C(k) - d(k)\frac{k^2}{4\pi G}\int \frac{d\eta}{z^2},$$
$$\varphi_\chi(k,\eta) = 4\pi G C(k)\frac{H}{a}\int z^2 d\eta + \frac{H}{a} d(k), \qquad (7)$$

where to the higher-order in the large-scale expansion each of the four solutions have $[1+\sum_{n=1,2,3,...}\tilde{c}_n(k|\eta|)^{2n}]$ factor with $\tilde{c}_n$ differing for the four cases. We *emphasize* the general nature of these solutions in the large-scale limit. These are exact solutions of the spatial curvature perturbation (φ) in the respective hypersurfaces (gauges) valid as long as the k^2 terms in eq. (4) are negligible; thus valid for general (time-varying) potential V. Similar general solutions exist for the fluid situation and even for the generalized gravity theories [13].

III. POWER-LAW EXPANSION

A field with an exponential potential supports power-law expansion/contraction of the scale factor [14]

$$a \propto |t|^p \propto |\eta|^{p/(1-p)}, \quad V = -\frac{p(1-3p)}{8\pi G} e^{-\sqrt{16\pi G/p}\phi},$$
$$H/\dot{\phi} = \sqrt{4\pi G p}. \qquad (8)$$

In the power-law case eq. (4) leads to Bessel equations for v and u with different orders. Using the quantization based on the action formulation in eq. (6), we have the exact mode function solutions ($p \neq 1$) [5]

$$\varphi_{\delta\phi k}(\eta) = \left|\frac{H}{\dot{\phi}}\right| \frac{\sqrt{\pi|\eta|}}{2a}\left[c_1(k)H^{(1)}_{\nu_v}(x) + c_2(k)H^{(2)}_{\nu_v}(x)\right],$$
$$\varphi_{\chi k}(\eta) = \frac{|H|\sqrt{\pi^2 G|\eta|}}{k\sqrt{p}}\left[c_1(k)H^{(1)}_{\nu_u}(x) + c_2(k)H^{(2)}_{\nu_u}(x)\right],$$
$$\nu_v \equiv \frac{3p-1}{2(p-1)}, \quad \nu_u \equiv \frac{p+1}{2(p-1)}, \qquad (9)$$

where $x \equiv k|\eta|$. The quantization condition implies $|c_2|^2 - |c_1|^2 = \pm 1$ depending on the sign of η, [5].

IV. MODE IDENTIFICATION

The Hankel functions can be expanded as [15]

$$H^{(1,2)}_\nu(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{x^2}{4}\right)^n \frac{1}{\sin\nu\pi}\left[\left(\frac{x}{2}\right)^\nu \frac{\pm i e^{\mp i\nu\pi}}{\Gamma(\nu+n+1)} + \left(\frac{x}{2}\right)^{-\nu}\frac{\mp i}{\Gamma(-\nu+n+1)}\right]. \qquad (10)$$

Notice that, in the small x limit, the first (second) term in the parenthesis dominates for $\nu < 0$ ($\nu > 0$). In eq. (7) the leading orders of the C-modes are time independent whereas the leading orders of the d-modes behave as $\varphi_{\delta\phi} \propto |\eta|^{2\nu_v}$ and $\varphi_\chi \propto |\eta|^{2\nu_u}$. Since

$$\varphi_{\delta\phi k} \propto |\eta|^{\nu_v} H^{(1,2)}_{\nu_v}(k|\eta|), \quad \varphi_{\chi k} \propto k^{-1}|\eta|^{\nu_u} H^{(1,2)}_{\nu_u}(k|\eta|), \qquad (11)$$

we can easily identify the first and the second terms in the parenthesis of eq. (10) as the d-mode and the C-mode, respectively. The power spectrum and the spectral index are defined as $\mathcal{P}_\varphi = \frac{k^3}{2\pi^2}|\varphi_k|^2$ and $n_S - 1 \equiv d\ln\mathcal{P}_\varphi/d\ln k$. The spectral indices for the C-modes of $\varphi_{\delta\phi}$ and φ_χ can be read as (we assume the simplest vacuum state choice)

$$(n_S - 1)_{\varphi_{\delta\phi},C} = (n_S - 1)_{\varphi_\chi,C} = \frac{2}{1-p}. \qquad (12)$$

Although not interesting (because it becomes transient in an expanding phase) the spectral indices for the d-modes are

$$(n_S - 1)_{\varphi_{\delta\phi},d} = \frac{4-6p}{1-p}, \quad (n_S - 1)_{\varphi_\chi,d} = -\frac{2p}{1-p}. \qquad (13)$$

Notice that the spectral indices of the C-mode *coincide* in both gauge conditions, whereas the ones for the d-mode show *strong gauge dependence*. This is easily understandable from the general solutions in eq. (7): in the power-law expansion case we have $\varphi_{\delta\phi}/\varphi_\chi = 1+p$ for the C-mode; the C-mode of $\varphi_{\delta\phi}$ remains constant even under the changing potential whereas φ_χ changes it value. Similarly, for the d-mode we have $\varphi_{\delta\phi}/\varphi_\chi = \frac{(p-1)^2}{3p-1}(k|\eta|)^2$, thus we have $(n_S-1)_{\varphi_\chi,d} = (n_S-1)_{\varphi_{\delta\phi},d} - 4$.

V. CONSEQUENCES

In an ekpyrotic scenario with $0 < p \ll 1$ ($\nu_v \simeq \frac{1}{2}$ and $\nu_u \simeq -\frac{1}{2}$) we have a very blue $n_S - 1 \simeq 2$ spectrum for the C-mode. Although $n_S - 1 \simeq 0$ for the d-mode of φ_χ we are not interested in the d-mode (incidentally, we have $n_S - 1 \simeq 4$ for the d-mode of $\varphi_{\delta\phi}$). Our point is that, although for φ_χ the d-mode is the dominating (and relatively growing) solution in the collapsing phase, our classification of the C- and d-modes is based on the *general* large-scale solutions in eq. (7); the large-scale conditions used to get these solutions are well met during the transition phase in the ekpyrotic scenario, thus as long as the linear perturbation theory remains valid we can rely on these solutions without any mixing between the two modes. Thus, claiming the scale-invariant spectrum based on the d-mode of φ_χ is *incorrect.* Also notice that the d-modes show strong gauge dependence: the d-mode of $\varphi_{\delta\phi}$ shows very blue spectrum. [Near singularity, the d-mode of φ_χ diverges more strongly compared with the ones in the other gauge conditions [13]. The strong divergence in the zero-shear gauge is known to be due to the strong curvature of the hypersurface (temporal gauge condition) [16]. According to Bardeen the behavior of φ_χ "overstates the physical strength of the singularity", [16].] The C-modes of both $\varphi_{\delta\phi}$ and φ_χ show the same blue spectra. The complete spectrum of the C-mode and the one for the tensor-type perturbation can be found in [5].

In a similar context, for $p = \frac{2}{3}$ ($\nu_v = -\frac{3}{2}$ and $\nu_u = -\frac{5}{2}$) we have $n_S - 1 = 0$ for the d-mode of $\varphi_{\delta\phi}$ (in this case we have $n_S - 1 = -4$ for the d-mode of φ_χ). Identifying this as another possibility for generating a scale-invariant spectrum attempted in [9] is *incorrect* for the same reason as in the ekpyrotic case; this was pointed out in [5]. The C-mode is the one we are interested, and we have $n_S - 1 = 6$, thus too blue. Yet another similar error was made in [10], now in the case of $p = \frac{1}{2}$. ($\nu_v = -\frac{1}{2}$ and $\nu_u = -\frac{3}{2}$). In this case we have $n_S - 1 = 4$ for the C-modes, and 2 for $\varphi_{\delta\phi}$ and -2 for φ_χ for the d-modes. Based on the zero-shear gauge authors of [10] claimed that the generated spectrum has $n_S = -1$ for $p = \frac{1}{2}$ and $n_S = 1$ for $p \simeq 0$ (ekpyrotic!), both of which are the ones for the d-mode of φ_χ, thus irrelevant for the final surviving (observationally relevant) spectrum.

If the linear perturbation theory is valid throughout, and the bounce is smooth and non-singular (see [7] for several examples) we could rely on solutions in eq. (7) as long as the large-scale conditions are met. In such a case, as emphasized in [5], we do not need to use matching conditions which, if we used properly, also give the same result [17,4,5,7,8]. Several possibilities to have (smooth and non-singular) bounce models were studied in [7]. In [7], using a toy bounce model based on an exotic matter with a negative energy density we have shown that the pre- and post-bounce results of the C-modes of $\varphi_{\delta\phi}$ and φ_χ show the same behaviors as the ones we studied in this work (which ignores the precise physics of the bounce), *independently* of the presence of the exotic matter (and the bounce itself) introduced to connect the collapsing and the expanding phases, see §V.C in [7].

From eq. (12) we can read that the only way to get a $n_S - 1 \simeq 0$ spectrum from the power-law expansion based on an exponential potential is to have $p \gg 1$ ($\nu_v \simeq \frac{3}{2}$ and $\nu_u \simeq \frac{1}{2}$) which is the ordinary power-law expansion *or* a damped collapsing phase. As pointed out in [5], in the latter case as the model approaches the bouncing phase the comoving scales shrink faster than the Hubble (dynamical) horizon. Thus, the large-scale condition can be violated near the bounce, and we cannot simply trace the perturbation through the bounce, see [7]. Lyth [6] has expounded that the linear perturbation theory *breaksdown inevitably* as the model approaches the singularity in a singular bounce (if such a bounce is possible at all, [18]).

Acknowledgments

We thank Robert Brandenberger for useful correspondences and Ewan Stewart for helpful discussions. This work was supported by Korea Research Foundation grants (KRF-2001-041-D00269).

[1] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, Phys. Rev. D **64**, 123522 (2001), hep-th/0103239; hep-th/0109050; P. J. Steinhardt, and N. Turok, hep-th/0111098.

[2] R. Durrer, hep-th/0112026.

[3] D. H. Lyth, Phys. Lett. B **524**, 1 (2002), hep-th/0106153.

[4] R. Brandenberger, and F. Finelli, JHEP **0111**, 056 (2001), hep-th/0109004.

[5] J. Hwang, Phys. Rev. D **65**, 063514 (2002), astro-ph/0109045.

[6] D. H. Lyth, Phys. Lett. B **526**, 173 (2002), hep-th/0110007.

[7] J. Hwang, and H. Noh, Phys. Rev. D, in press (2002), astro-ph/0112079.

[8] J. Martin, P. Peter, N. Pinto-Neto, D. J. Schwarz, hep-th/0112128.

[9] F. Finelli, and R. Brandenberger, hep-th/0112249.

[10] P. Peter, and N. Pinto-Neto, hep-th/0203013.

[11] J. M. Bardeen, in: *Cosmology and particle physics*, eds., L. Fang and A. Zee, (Gordon and Breach, London, 1988), 1; J. Hwang, Astrophys. J. **375**, 443 (1991); J. Hwang, and H. Noh, Phys. Rev. D **65**, 023512 (2002), astro-ph/0102005.

[12] V. N. Lukash, Sov. Phys. JETP **52**, 807 (1980); Sov. Phys. JETP Lett. **31**, 596 (1980); V. F. Mukhanov, *ib id.* **41**, 493 (1985); Sov. Phys. JETP **68**, 1297 (1988); V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).

[13] J. Hwang, Astrophys. J. **415**, 486 (1993); *ib id.* **427**, 542 (1994); J. Hwang and H. Noh, Phys. Rev. D **54**, 1460 (1996).

[14] F. Lucchin, and S. Matarrese, Phys. Rev. D **32**, 1316 (1985).

[15] M. Abramowitz, and I. A. Stegun, edited, *Handbook of mathematical functions*, (Dover Publications, New York, 1972).

[16] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[17] J. Hwang, and E.T. Vishniac, Astrophys. J. **382**, 363 (1991).

[18] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt, and N. Turok, hep-th/0108187; G. Felder, A. Frolov, L. Kofman, and A. Linde, hep-th/0202017.